An Interpretation of the Stability of a Turbulent Shear Flow Near a Rough Wall

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A conventional, small perturbation, stability analysis has been applied to a fully developed turbulent shear flow in a parallel duct with rough walls. This is an attempt to detect the inherent state of flow stability to quasi-regular disturbances emanating from the surface roughness elements. The surface roughness is represented by the usual roughness Reynolds number; it is fed into the analysis through an assumed mean velocity profile valid between the viscous sublayer and the inner (wall) region. An eddy viscosity model is used to secure the equation closure and the final equation for the perturbation amplitude has been solved numerically using the techniques developed for the Orr–Sommerfeld equation.

Within the domain of realistic flow conditions, and for a range of surface roughness amplitudes, a local minimum of stability in terms of the longitudinal wave number has been found. However, it is not implied that it is a minimum minimorum, as only a limited range of surface roughnesses has been tried.

NOTATION

a	width of two-dimensional parallel channel
C	complex wave propagation velocity $(=C_r + iC_i)$
C_i	damping/amplification factor (imaginary component)
C_r	wave propagation velocity in longitudinal direction (real)
h	length scale of surface roughness
h_*	surface roughness Reynolds number $(=hu_0/\nu)$
I	functional
i	imaginary number $(=\sqrt{-1})$
k	wave number
l	length scale $(\gg l')$
l'	length scale pertaining to q' $(\ll l)$
$\tilde{p}$	pressure component corresponding to $\tilde{q}$
Q	general flow field quantity $(=\bar{Q} + \tilde{q} + q')$
$\bar{Q}$	any mean flow component
q'	any random turbulent component
$\tilde{q}$	any oscillating flow component of relatively low frequency
Re	flow Reynolds number $(=aV/\nu)$
Re_x	flow Reynolds number of a boundary layer
T	period of low frequency oscillations
t	time
U_*	nondimensional velocity in wall shear flows $(=\bar{U}/u_0)$
$\bar{U}_k$	velocity component corresponding to $\bar{Q}$ (k = 1, 2, 3 = cartesian tensor index)
u_i', u_k'	velocity components corresponding to q' ((i = 1, 2, 3; k = 1, 2, 3) = cartesian indices)
u_0	wall friction velocity $\{=\sqrt{(\tau_0/\rho)}\}$
$\tilde{u}_i$, $\tilde{u}_k$	velocity components corresponding to $\tilde{q}$ ((i = 1, 2, 3; k = 1, 2, 3) = cartesian indices)
V	average velocity in channel
v	amplitude of transverse oscillating velocity
x	longitudinal coordinate $(=x_1)$
x_i, x_k	cartesian coordinates ((i = 1, 2, 3; k = 1, 2, 3) = cartesian indices)
$\mathbf{x}$	position vector
y	transverse coordinate $(=x_2)$
y_*	nondimensional distance from wall $(=yu_0/\nu)$
y_{v*}	edge of viscous sublayer (in terms of y_*)
κ	von Kármán constant $(\approx 0{\cdot}41)$
ν	kinematic viscosity of fluid
ν_T	eddy viscosity of turbulent motion
ρ	fluid density
τ_0	wall friction stress
ω	complex frequency of oscillations $(=kC)$
$\bar{}$	(overhead bar) time/space average
$\langle\ \rangle$	phase average
$'$	differentiation with respect to y $(=\mathrm{d}/\mathrm{d}y)$

1 INTRODUCTION

Practical aspects of the effects of surface roughness (in particular of marine nature) have been studied by many and recently in some greater depth by the Liverpool University Group (e.g., (**1**), (**2**)). Apart from the problem of the local wall friction (due to the form and molecular drag components), being the outcome of a complex interaction between the turbulent and mean flow fields, it is thought that the surface roughness may affect the general stability of the resultant turbulence. Therefore, an analysis, based on already tried techniques, could prove 'instrumental' in studying the problem generally. It is tentatively explored here.

Hussain and Reynolds (**3**) and (**4**) investigated experimentally a two-dimensional turbulent channel flow with a periodic ('organized wave') component of frequency 25–100 Hz introduced artificially into the flow. All turbulent field quantities (velocities, pressure, etc.) in such a situation comprise three components:

(1) a mean flow component, $\bar{Q}$;
(2) a relatively large oscillating component, $\tilde{q}$, of scale l;
(3) a random turbulent component, q', of scale $l' \ll l$.

Hence for a general field quantity one writes

$$Q = \bar{Q} + \tilde{q} + q' \qquad (1)$$

The same model is postulated here for turbulent shear flows near a rough surface, the roughness profile of which contains a wavy, or nearly wavy, nonseparating low frequency component creating these relatively large periodic oscillations.

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2 AVERAGED BASIC EQUATIONS

Invoking the definition of 'phase average'

$$\langle q \rangle = \lim_{n\to\infty} \frac{1}{n} \sum^{n} q(\mathbf{x}, t + nT) \tag{2}$$

where $\mathbf{x}$ is the position vector, t is the time, and T is the period of oscillations, Hussain and Reynolds demonstrate that

$$\langle Q \rangle = \bar{Q} + \tilde{q} \tag{3}$$

This averaging process eliminates the random turbulent component (i.e., filters out q' from the total signal).

Subsequently, Reynolds and Hussain (**5**) derived rigorously the pertaining dynamic equations of motion by applying the transformation (1) in the Navier–Stokes equations along with the usual time averaging followed by the phase averaging (eq. (2)). Their result, in cartesian tensor notation, reads as follows

$$\frac{\partial \tilde{u}_i}{\partial t} + \bar{U}_k \frac{\partial \tilde{u}_i}{\partial x_k} + \tilde{u}_k \frac{\partial \bar{U}}{\partial x_k} = -\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_k^2} + \frac{\partial}{\partial x_k} \{(\overline{\tilde{u}_i \tilde{u}_k} - \tilde{u}_i \tilde{u}_k) - (\langle \tilde{u}_i' \tilde{u}_k' \rangle - \overline{u_i' u_k'})\} \tag{4}$$

together with the pertaining flow continuity equations

$$\frac{\partial \bar{U}_i}{\partial x_i} = \frac{\partial \tilde{u}_i}{\partial x_i} = \frac{\partial u_i'}{\partial x_i} = 0 \tag{5}$$

3 LINEARIZED EQUATIONS OF STABILITY AND MODEL OF TURBULENCE

Now in the spirit of the classical theory of hydrodynamic stability, these equations can be linearized by applying the small perturbation technique to the dynamic equations of motion. As usual, they are then reduced to the parallel mean flow for simplicity. However, one must bear in mind that, unlike the equations for laminar flows, the last term on the right of eq. (4) renders it indeterminate unless some closure conditions are found which can be drawn from the semi-empirical theories of turbulence. This will be dealt with later.

A standard expression for $\tilde{q}$, treating it as a small perturbation (cf. e.g., Betchov and Criminale (**6**)) for pure two-dimensional oscillations, is

$$\tilde{q} = \frac{1}{2} q(y) \exp\{ik(x - Ct)\} + \text{conjugate part} \tag{6}$$

where $i = \sqrt{-1}$; k is the wave number in the direction of x; $C = C_r + iC_i$ which is the wave propagation velocity; C_r is the wave propagation velocity in x-direction; C_i is the damping ($C_i < 0$)/amplification ($C_i > 0$) factor.

The circular frequency of oscillation is $\omega = kC$. Restricting the present case to homogeneously distributed surface roughnesses (waviness), there is no spatial development of the perturbations but merely a temporal one.

Expressing the oscillating terms in eqs (4) and (5) in terms of eq. (6) and eliminating $\tilde{p}$, a single, fourth-order, ordinary differential equation for the transverse perturbation amplitude, v, emerges. It is the Orr–Sommerfeld equation supplemented by the oscillating Reynolds stress terms.

The authors of reference (**5**) managed to account for the extra terms using Lumley's (7) turbulent stress/strain relationship which involves the eddy viscosity ν_T of the basic shear flow. Their resultant equation for v is

$$(\bar{U} - C)(v'' - k^2 v) - \bar{U}'' v + \frac{i}{k}\{(\nu + \nu_T)(v'''' - 2k^2 v'' + k^4 v) + 2\nu_T(v''' - k^2 v') + \nu_T''(v'' + k^2 v)\} = 0 \tag{7}$$

where $(') \equiv \mathrm{d}/\mathrm{d}y$.

Of course, eq. (7) converts immediately to the pure Orr–Sommerfeld equation for $\nu_T = 0$.

In order to solve eq. (7) one requires information about $\bar{U}(y)$ and $\nu_T(y)$. As a 'pilot' exercise, it has been decided to calculate the two-dimensional, fully developed turbulent flow in a channel with rough walls a apart. The respective boundary conditions are

$$v(0) = v'(0) = v(a) = v'(a) = 0 \tag{8}$$

Instead of the two latter boundary conditions it is numerically more appropriate to employ the conditions of symmetry at the centreline, i.e., $v(a/2) = v''(a/2) = 0$.

4 MEAN FLOW AND EDDY VISCOSITY CLOSURE

With regard to the closure input, two variants have been tried.

(1) The *quasi-laminar* approach—by which one feeds information about turbulence into the system solely through the mean velocity field $\bar{U}(y)$. Here it appears particularly appropriate to use the Hudimoto (**8**) expression which in 'wall' variables $U_* = \bar{U}/u_0$ and $y_* = yu_0/\nu$ (u_0 is the wall friction velocity) reads

$$U_* = y_{v_*} + \frac{1}{\kappa} \ln [2\kappa(y_* - y_{v_*}) + \sqrt{\{1 + 4\kappa^2(y_* - y_{v_*})^2\}}] + \frac{1 - \sqrt{\{1 + 4\kappa^2(y_* - y_{v_*})^2\}}}{2\kappa^2(y_* - y_{v*})} \tag{9}$$

where y_{v*} = thickness of viscous sublayer.The effect of eddy viscosity is ignored ($\nu_T = 0$) in this case.

(2) The *eddy-viscosity* approach in which, in addition to the same $\bar{U}(y)$ input (as in (1)), one can use

$$\frac{\nu_T}{\nu} = 1 + \kappa^2(y_* - y_{v*})^2 \frac{\mathrm{d}U_*}{\mathrm{d}y_*} \tag{10}$$

which is fully consistent with eq. (9).

In both cases, Millionshchikov's (**9**) hypothesis of the geometrically additive nature of the surface roughness effects has been applied. This necessitates replacing $(y_* - y_{v*})$ with $(y_* + h_* - 2y_{v*})$, in eqs. (9) and (10), where h_* is the roughness Reynolds number.

Equations (9) and (10) are a useful device, particularly so, because both y_{v*} and h_* may be fed arbitrarily into them. This freedom permits the stability tests to be parametrically related to the state of surface roughness.

5 SOLUTION AND VARIATIONAL INTERPRETATION OF RESULTS

The numerical solution of the problem can be carried out quite effectively and efficiently by means of a method based on the Riccati transformation technique described by Jankowski, Takeuchi, and Gerstin (**10**).

Actual calculations have been performed† on a 1906S computer assuming the flow Reynolds number $Re = aV/\nu = 10^4$ (where V is the channel average velocity) and $y_{\nu*} = 5$, for both cases: $\nu_T = 0$ and $\nu_T \neq 0$.

Omitting the bulk of the working results pertaining to the computations, reference is made here to Goldshtik's (**11**) formulation of the principle of maximum stability of turbulent shear flows in response to external perturbations—in this case to the low frequency effects of the wall roughness.

According to Goldshtik, the stability measure is a functional in relation to the class of velocity profiles with variational coefficients such that

$$I_1 = \sup_k C_i(k) \tag{11}$$

since C_i depends on k. Function $C_i(k)$ may have more than one local maximum; the absolute maximum (cf. eq. (11)) must be found in order to intercept the most vulnerable stability case. Naturally, the most stable is the flow for which I_1 assumes the lowest negative value (absolute minimum).

Unfortunately, functional I_1 is not always well behaved with respect to convenient flow parameters. On account of this, Goldshtik and Kutateladze (**12**) invented another functional. It is less susceptible to such weaknesses as it is based on the concept of kinetic energy of oscillations ($\propto \exp(-2kC_i t)$, for negative C_i). The functional reads

$$I_{2,N} = \iiint_0^\infty \tilde{v}^2 \, dk \, dt \, dx \sim -\int_0^\infty \frac{d(\ln k)}{C_{i,N}} \tag{12}$$

where N designates the individual member of the class of flows (in the present case $N \equiv h_*$). For stable flows $I_{2,N}$ is finite. Wherever $C_{i,N}$ approaches zero instability sets in and correspondingly $I_{2,N} \to \infty$.

Intuitive interpretation of eq. (6), or more rigorously, an asymptotic expansion of the solution of eq. (7) for large and small values of k, shows that principal contributions to $I_{2,N}$ come from the short and long wavelengths of the oscillatory perturbations. This is independent of $\bar{U}(y)$ but dependent on the flow Reynolds number.

Taking a class of turbulent shear flows dependent on a flow parameter N (for example, N might be the exponent of the power-law approximation in turbulent boundary layers and fully developed turbulent pipe flows) one computes the plots $C_{i,N}(k)$-curves. Incidentally, in an exercise such as that referred to in parentheses above it would be quite in order not to use anything more than $\nu_T =$ constant, or even $\nu_T = 0$, bearing in mind the crudeness of the power-law approximation in the first place.

Interesting calculations were carried out in reference (**12**) for a turbulent shear flow near a solid (smooth) wall with the mean velocity distribution established from the eddy viscosity relationship: $\nu_T = \nu + \kappa u_0 y$, but ignoring the stress/strain relationship terms in eq. (7), i.e., solving it as an Orr–Sommerfeld equation. The final result

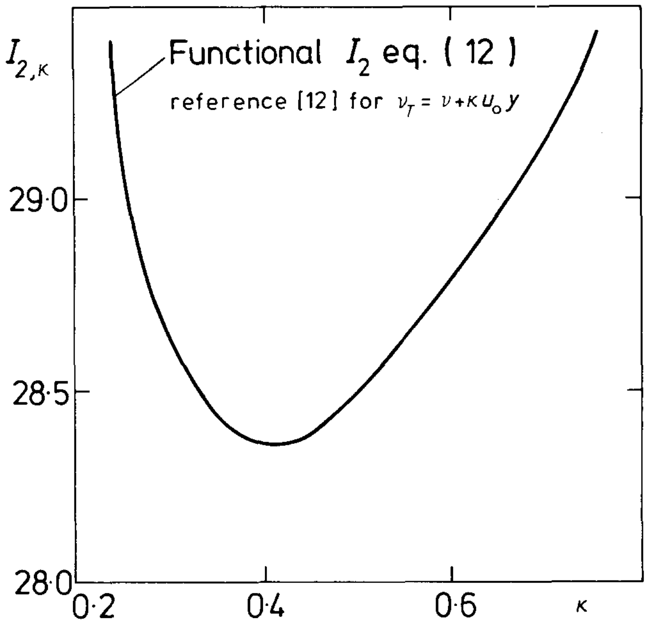

Fig. 1. Natural 'stability' functional of von Kármán constant for turbulent wall shear flow (after Goldshtik and Kutateladze)

plotted in Fig. 1 shows the most stable flow at $\kappa \approx 0{\cdot}4$—which is in accord with many experimental findings and thus confirms the universally accepted value. This is mentioned here only parenthetically to indicate the role the stability analysis can play in theoretical investigations of the wall turbulence. Of course, there is no other connection between Goldshtik and Kutateladze's problem and the present wall roughness stability effect.

6 DISCUSSION AND CONCLUSIONS

The above Russian variational rationale has been finally employed in presenting the calculations evolved for the purpose of this article, as described earlier on in subsection 5. The parameter identified with N is the roughness Reynolds number h_* and the 'specific' presentation of the ensuing results is shown in Fig. 2. Two lots of curves are presented here: (*a*) continuous curves representing solutions of the complete eq. (7) with $\nu_T \neq 0$, for values of $h_* = 0{\cdot}1$, 1, 10, 10^2, and 10^3, and (*b*) broken curves for values of $h_* = 1$, 10, and 10^2, all for $\nu_T = 0$.

The corresponding curves for the two eddy viscosity cases are not too different and certainly show the same trend. It appears that, for the limited data available, the least stable is the flow condition at $h_* \approx 1$. The absolute maximum of C_i/u_0, apart from the terminal limits, $k \to 0$ and $k \to \infty$, occurs at $ka \approx 11$ for $\nu_T \neq 0$; for $\nu_T = 0$ it takes place, albeit less distinctly, at $ka \approx 10$. With increasing and decreasing roughness Reynolds number the instability diminishes. This is not surprising as the fully rough regime promotes naturally turbulent diffusion near the wall and also generates additional relatively high frequency turbulence. These two agents are likely to be responsible for the noticeable stabilization of the turbulent flow for $h_* = 10$ and higher.

The problem has not been without numerical difficulties, specifically, the convergence in the numerical treatment of eq. (7) combined with eqs. (8)–(11) has been troublesome (although not so much for $\nu_T = 0$). The

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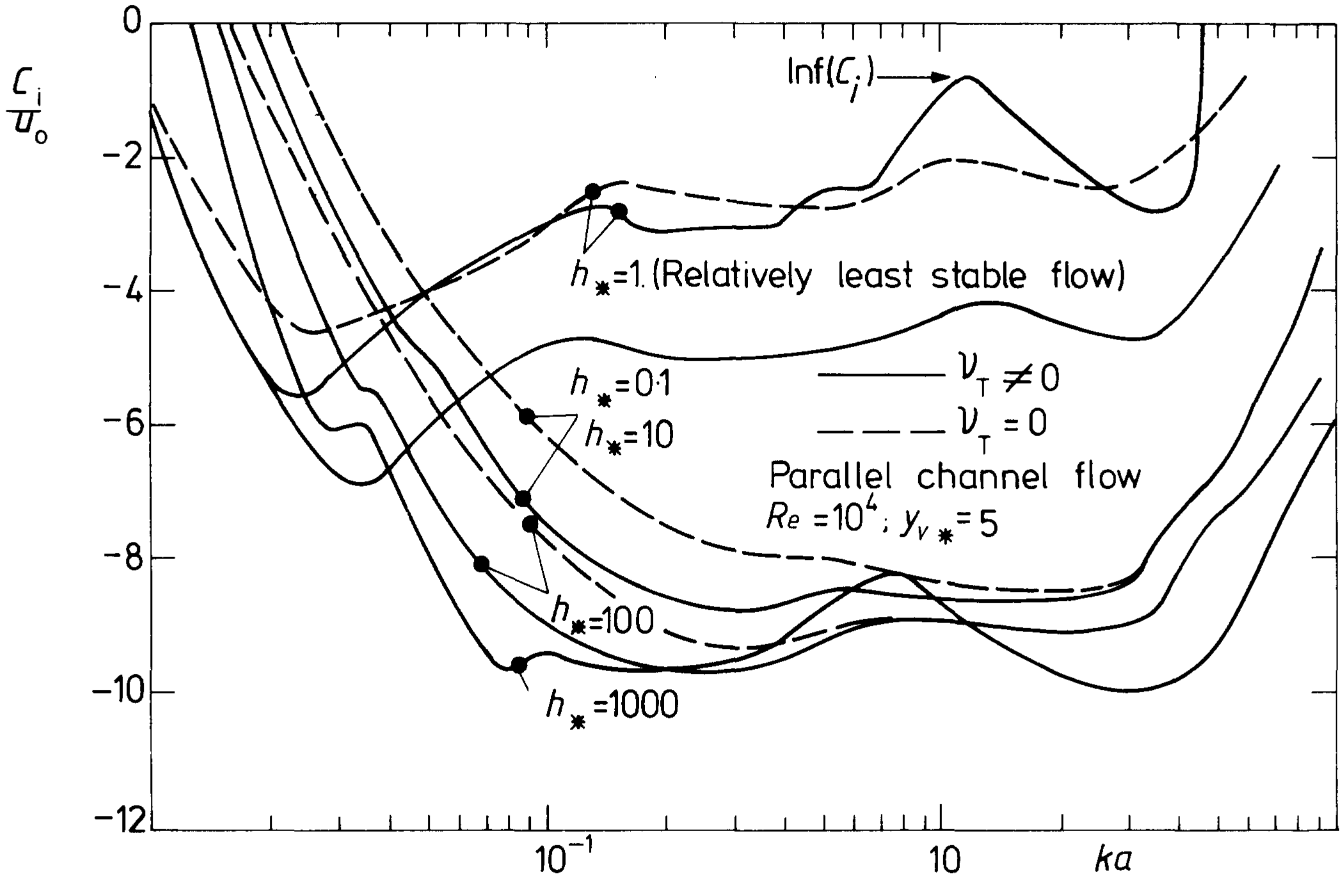

Fig. 2. Damping factor for fully developed turbulent duct flow as function of wave number and parametrically dependent on magnitude of surface roughness

similarity of the $\nu_T = 0$ and $\nu_T \neq 0$ curves in Fig. 2 seems to indicate that the $\nu_T \neq 0$ curves are probably numerically acceptable. This speculative remark is based on the fact that with $\nu_T = 0$, eq. (7) becomes the classical Orr–Sommerfeld equation which had been solved over the years by several independent numerical schemes, e.g., references (**3**) and (**5**) and following analytical solutions of the theory of hydrodynamic stability. The present solution is not much different from such numerical schemes (cf. (reference **10**)). With $\nu_T \neq 0$ the modified Orr–Sommerfeld equation could have assumed a less stable nature. There is not much evidence for that and the closeness of $\nu_T = 0$ and $\nu_T \neq 0$ solutions seems to suggest that the 'turbulent-cum-fluctuating' stress tensor may have merely a corrective influence—in the present case, anyway.

It would be very interesting, and indeed desirable, to see if the surface roughness, which renders the fully turbulent wall region least stable (in this case $h_* = 1$), has any profound effect on the development, and possible separation, of the turbulent boundary layer. What is required here is a series of repetitive calculations of an arbitrary boundary layer, under an adverse pressure gradient, with all external parameters artificially fixed, but with variable h_*. A correlation should be sought between Inf $[C_i(h_*, Re)]$ and some unusual development feature(s) of the boundary layer, e.g., accelerated separation, sudden variation in the wall friction, etc. A method of calculating turbulent boundary layers on rough walls, recently evolved by A. J. Musker (**2**), could be instrumental in such an exercise.

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